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Stochastic Error Modeling of Beamformer Output for Arrays with Directive Elements

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PREFACE

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The error analysis of a two-dimensional passive array (conformal or planar) with directive element responses is developed. The analysis is presented for failures (both compensated and noncompensated), element phase errors, and element amplitude errors. The combination of these various mechanisms and their effect on mean beampattern power is given analytically. The report concludes with numerical examples of the errors, as well as with the analytical derivation of various physical error mechanisms in terms of amplitude and phase error. Although developed for conformal arrays of velocity sensors, the mathematical derivations in the analysis are general enough to apply to a wide variety of passive arrays.

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STOCHASTIC ERROR MODELING OF BEAMFORMER OUTPUT FOR ARRAYS WITH DIRECTIVE ELEMENTS

1. INTRODUCTION

The fundamental design goal of a passive array is to detect weak signals over a broad band of frequencies in the presence of noise that is due to structures, water flow, the surrounding ambient environment, and extraneous targets. The array design is accomplished by matching some spatial filtering and signal filtering (commonly known as beamforming) to the expected noise and signal characteristics. The broadband beamforming is achieved through careful application of a time delay to each element of the array (or phase delay in the frequency domain), which creates a passive spatial/temporal filter for the expected signal and noise characteristics. The elements in the final array are expected to behave as modeled mathematically (i.e., the performance of the array elements will match the mathematical model used for the beamformer).

When arrays are manufactured as integral components of structures (as in conformal/hull arrays), isolating the array components from inherent manufacturing uncertainties becomes a problem that can be addressed through *array tolerance analysis*. This problem has historically been treated both in textbooks[1,2] (general arrays) and in technical reports[3,4] (sonar arrays). In this report, the array tolerance analysis is developed with two new approaches to the previous research.

The first approach involves the addition of the effects of directional sensors on the curved surface of the array, which is especially desirable for a conformal application because the sensors can then reject much of the structural noise through their directive response. However, the added directivity makes the phased-element addition task of the beamformer very sensitive to certain errors.

The second area addressed in this report concerns the handling of failures. Failures are considered in the traditional sense, as well as for the case of failure compensation through companion element blanking. Both techniques disregard the response from certain companion elements (to the failed element) in the array to preserve specific array symmetries in the presence of failures. Two basic types are considered: diagonal element failure compensation and beampattern symmetry failure compensation. Diagonal element failure compensation is an approach to array element failures that effectively zeros-out elements that are geometrically diagonal (relative to the acoustic center) to the failed element, thus maintaining the array's acoustic center. Beampattern symmetry failure compensation effectively zeros-out those elements that are the same distance from the acoustic center as the failed sensor, thus maintaining the array symmetry in either the horizontal or vertical beampattern. Other failure compensation techniques are not treated here because they are beyond the scope of this effort.

For the cases considered, it is assumed that deterministic effects not dependent on statistical parameters are included in the governing deterministic beamforming expression, where they are evaluated computationally. The required statistical array tolerance effects are shown in a progressive stochastic analysis.

Sections 2 and 3 of this report present some mathematical methods for array beamforming and statistical analysis that are then used throughout the analysis. The error/tolerance analysis begins in section 4 with the case of array failures, both compensated and uncompensated. Section 5 presents the next stochastic addition, which treats element phase variations, and section 6 uses element amplitude variations to complete the stochastic analysis. The important equations from the error analysis are summarized and some computational examples are shown in section 7. Finally, in section 8, examples of physically relevant errors and their effects due to phase or amplitude variations are described.

2. ARRAY BEAMFORMER

A general beamforming expression in the frequency domain for an array of M elements is given as

$$V_0(\omega) = \sum_{m=1}^{M} w_m \, d_m(\vec{\zeta}_i) \exp\left[i \, (\vec{k}_i - \vec{k}_s) \cdot \vec{x}_m\right], \tag{1}$$

where $\vec{k}_i = \vec{\zeta}_i\omega/c$ and $\vec{k}_s = \vec{\zeta}_s\omega/c$ are the wavevectors in the incident and steer (or look) directions, respectively. In this notation, $\vec{\zeta}$ represents a unit vector in three-dimensional space (hence, $||\vec{\zeta}|| = 1$). For completeness, the time-domain beamform output associated with equation (1) is given by

$$v_0(t) = \sum_{m=1}^{M} w_m \, d_m(\vec{\zeta}_i) \, u_m\left(t - \frac{(\vec{\zeta}_i - \vec{\zeta}_s) \cdot \vec{x}_m}{c}\right), \tag{2}$$

where $u_m(t)$ is the voltage output waveform of the m^{th} hydrophone. These two expressions are related through a standard Fourier transform.

The beamforming expression in equation (1) includes an element directivity term, $d_m(\vec{\zeta}_i)$, which depends on the incident wave direction $\vec{\zeta}_i$. For an example of this directivity term, it is assumed that the element directionality is given by a cosine function with perfect baffling as in a notional velocity sensor; hence,

$$d_m(\vec{\zeta}_i) = \max\{\vec{\zeta}_i \cdot \vec{n}_m, 0\}, \tag{3}$$

where $\vec{n}_m$ is the unit normal to the m^{th} array element. Because the dot product of two unit vectors is equal to the cosine of the angle between them, this expression represents a cosine directionality. Also seen in this equation is the parametric dependence of element directivity on the element normal (or orientation). Thus, it is shown that any variations in element orientation will directly affect element amplitude through the directivity expression.

3. STATISTICAL QUANTITIES OF INTEREST

The general beamformer expression is written as a sum of complex variables ν_m in the form

$$
\begin{aligned}
V(\omega) &= \sum_{m=1}^{M} w_m \, d_m(\vec{\zeta}_i) \exp\left[i \, (\vec{k}_i - \vec{k}_s) \cdot \vec{x}_m \right] , \\
&= \sum \nu_m ,
\end{aligned}
\tag{4}
$$

where the unmarked dependencies are all implied. In the context of this analysis, the above expression is a sum of complex random variables ν_m that represent the output of a random process. The quantity of interest when processing this beamformer output is the *beamformer output power spectrum*

$$
\begin{aligned}
|V(\omega)|^2 &= \left(\sum \nu_m\right)^* \left(\sum \nu_m\right) , \\
&= \sum_m \sum_{m'} \nu_{m'}^* \nu_m ,
\end{aligned}
\tag{5}
$$

where $(\cdot)^*$ represents a complex conjugate.

The statistics of $|V(\omega)|^2$ are important to the analysis of tolerance errors during array manufacturing, as well as to the effect of these errors on array performance. In this study, the statistics derived denote the expected value of the random output variable. Throughout the sequel, the notation $E\{x\}$ is used to represent the expected value and $Var\{x\}$ is used to represent the variance of a complex random variable x. While the variance is not derived explicitly, the techniques and simplifications developed are easily applied to derive expressions for variance. It is important to note that

$$
E\{f(x)\} \neq f(E\{x\}) , \tag{6}
$$

with the exception being for the linear function $f(x)$, so that, in general,

$$
E\{|V(\omega)|^2\} \neq |E\{V(\omega)\}|^2 . \tag{7}
$$

Thus, a simplistic analysis provides no useful results.

For the statistics on the beamformer output power spectrum, the results of Nuttall[4] are applied as shown below. The expected value of the power (magnitude squared) of a sum of complex variables gives

$$
\begin{aligned}
E\{|V(\omega)|^2\} &= E\{|\sum \nu_m|^2\} , \\
&= \sum \left(E\{|\nu_m|^2\} - |E\{\nu_m\}|^2 \right) + |E\{\sum \nu_m\}|^2 ,
\end{aligned}
\tag{8}
$$

which will be used throughout the sequel. For notational convenience, the implied dependence on ω is omitted, except for cases where it serves to clarify the narrative. Nuttall also shows that the variance of the power of a sum of complex variables can be written as

$$\begin{aligned} Var\{|V(\omega)|^2\} &= E\{\left(|V|^2 - E\{|V|^2\}\right)^2\}, \\ &= E\{|V|^4\} - \left(E\{|V|^2\}\right)^2, \\ &= E\{|\sum \nu_m|^4\} - \left(E\{|V|^2\}\right)^2, \\ &= \sum_{k\ell mn} E\{\nu_k \nu_\ell^* \nu_m \nu_n^*\} - \left(E\{|V|^2\}\right)^2, \end{aligned} \tag{9}$$

where the second term in the last expression is the square of the previously derived mean. The fourth-order sum in this equation is very complicated and, for that reason, cannot be written out here for the general case. In evaluating the fourth-order sum, Nuttall[4] shows that only five statistical quantities are required:

$$E\{\nu_m\},\ E\{|\nu_m|^2\},\ E\{\nu_m^2\},\ E\{|\nu_m|^2\nu_m\},\ \text{and}\ E\{|\nu_m|^4\}. \tag{10}$$

Although, in the analysis that follows, only the mean value is determined, the interested reader can use these techniques to determine values of the above quantities. The values can then be applied to the expressions found in Nuttall[4] and evaluated numerically. The expressions that combine these terms into the fourth-order sum in equation (9) are also given in reference 4.

4. ELEMENT FAILURES

The first stochastic variation under consideration is element failure. For array failure analysis, a binomial random variable f_m is used to model the failure state of the m^{th} element. The two states of this binomial variable are an element failure, $f_m = 1$, and a working element, $f_m = 0$. The failure variable impacts the m^{th} term in the beamformer sum of equation (1) by a multiplication factor of $1 - f_m$. The probability density function or probability distribution of the binomial random variable f_m is given by

$$p\,(f_m) = Q\delta(f_m - 1) + (1 - Q)\delta(f_m)\;, \tag{11}$$

where $\delta(\cdot)$ is the Dirac delta and Q is the individual element failure rate (i.e., an element is failed with probability Q). For the distribution given above, the stochastic moments of the quantity $1 - f_m$ are given by

$$E\{(1 - f_m)^k\} = \int (1 - f_m)^k p\,(f_m) df_m = 1 - Q\,, \qquad \forall\, k > 0\,, \tag{12}$$

which provides all the moments necessary for a stochastic analysis of the element failure for *independent* failure events within the beamformer expression.

When element failure compensation techniques are applied, element failures become statistically *dependent* events. From the beamformer output perspective, this means that elements always fail in pairs (i.e., if one element fails, the corresponding zeroed-out element does not contribute to the beamformer output and has thus "failed"). Because these pairs are related through some geometric symmetry (either left-right, up-down, or diagonal, depending on the failure compensation scheme), there is a strong geometrical dependence on the failure, which must be accounted for in the mathematical model.

To maintain generality, it is assumed that the array elements are numbered such that the $(M + 1 - m)^{th}$ element is symmetrically opposing the m^{th} element, in the manner corresponding to the element failure compensation scheme employed. Now the beamforming expression (1) is rewritten as (for convenience, it is assumed that M is even and odd M is a simple extension)

$$V_0(\omega) = \sum_{m=1}^{M/2} \left[z_m + z_{\hat{m}}\right]\,, \tag{13}$$

with z_m representing the nominal beamformer contribution of the m^{th} element as

$$z_m = w_m\, d_m(\vec{\zeta}_i) \exp\left[i\,(\vec{k}_i - \vec{k}_s) \cdot \vec{x}_m\right] \tag{14}$$

and $\hat{m} = M + 1 - m$. The failure of independent elements is governed by the failure variable f_m

defined above, and accounting for compensation, the failure enters the beamforming expression as

$$\begin{aligned} V_1(\omega) &= \sum_{m=1}^{M/2} [(1-f_m)(1-f_{\hat{m}})z_m] + [(1-f_{\hat{m}})(1-f_m)z_{\hat{m}}] \,, \\ &= \sum_{m=1}^{M/2} (1-f_m)(1-f_{\hat{m}}) \left[z_m + z_{\hat{m}} \right] \,, \end{aligned} \tag{15}$$

where the subscript 1 on $V_1(\omega)$ is used to represent the case of element failures. This expression shows that if either the m^{th} or the $\hat{m}^{th}$ element fails (either $f_m = 1$ or $f_{\hat{m}} = 1$), neither element contributes to the beamformer output $V_1(\omega)$.

In equation (15), the product of failures is the only stochastic quantity that appears. This product is denoted as

$$\beta_m = (1-f_m)(1-f_{\hat{m}}), \tag{16}$$

such that

$$V_1(\omega) = \sum_{m=1}^{M/2} \beta_m (z_m + z_{\hat{m}}) \,, \tag{17}$$

where the bivariate random variables $\{\beta_m\}_{m=1}^{M/2}$ are independent of one another. The stochastic moments of this random variable are given by

$$E\{\beta_m^k\} = \int\int \beta_m^k p\,(f_m) p\,(f_{\hat{m}}) df_m df_{\hat{m}} = (1-Q)^2 \,, \qquad \forall\, k > 0 \,, \tag{18}$$

which is used throughout the study. With these moments defined, the expected value of the beamformer output power spectrum is evaluated as

$$E\{|V_1|^2\} = \sum_{m=1}^{M/2} \left(E\{|\chi_m|^2\} - |E\{\chi_m\}|^2 \right) \; + \; |E\{V_1\}|^2 \,, \tag{19}$$

where

$$\chi_m = \beta_m (z_m + z_{\hat{m}})$$

and a dependence on ω is assumed.

The first term in equation (19) is given by

$$\begin{aligned} E\{|\chi_m|^2\} &= E\{|\beta_m (z_m + z_{\hat{m}})|^2\} \,, \\ &= E\{\beta_m^2\} \cdot |z_m + z_{\hat{m}}|^2 \,, \\ &= (1-Q)^2 \cdot |z_m + z_{\hat{m}}|^2 \,; \end{aligned} \tag{20}$$

the second term is given by

$$
\begin{aligned}
|E\{\chi_m\}|^2 &= |E\{\beta_m(z_m + z_{\hat{m}})\}|^2 , \\
&= |E\{\beta_m\}|^2 \cdot |(z_m + z_{\hat{m}})|^2 , \\
&= (1-Q)^4 \cdot |z_m + z_{\hat{m}}|^2 ;
\end{aligned} \tag{21}
$$

and the third and final term is given by

$$
\begin{aligned}
|E\{V_1\}|^2 &= |E\{\textstyle\sum \beta_m(z_m + z_{\hat{m}})\}|^2 , \\
&= |\textstyle\sum E\{\beta_m\}(z_m + z_{\hat{m}})|^2 , \\
&= |\textstyle\sum (1-Q)^2(z_m + z_{\hat{m}})|^2 , \\
&= (1-Q)^4 \cdot |\textstyle\sum (z_m + z_{\hat{m}})|^2 , \\
&= (1-Q)^4 \cdot |V_0|^2 .
\end{aligned} \tag{22}
$$

Combining these terms back into equation (19) yields

$$
E\{|V_1|^2\} = [(1-Q)^2 - (1-Q)^4] \sum_{m=1}^{M/2} |z_m + z_{\hat{m}}|^2 \; + \; (1-Q)^4 \, |V_0|^2 . \tag{23}
$$

It is noted that an element failure rate Q of zero produces

$$
E\{|V_1|^2\} \;\rightarrow\; |V_0|^2 , \qquad \text{as} \qquad Q \rightarrow 0 , \tag{24}
$$

as expected.

The case of uncompensated failures is formulated in the same way to arrive at

$$
\begin{aligned}
E\{|V_{1,uncomp}|^2\} &= [(1-Q) - (1-Q)^2] \sum_{m=1}^{M/2} \left(|z_m|^2 + |z_{\hat{m}}|^2\right) \; + \; (1-Q)^2 \, |V_0|^2 , \\
&= [(1-Q) - (1-Q)^2] \sum_{m=1}^{M} w_m^2 \, d_m^2(\vec{\zeta}_i) \; + \; (1-Q)^2 \, |V_0|^2 ,
\end{aligned} \tag{25}
$$

which differs greatly from equation (23). Thus, the use of statistics without the geometrical considerations of compensation (such as doubling Q) is unwise and would yield highly questionable results. The uncompensated failure case is obtained from the compensated failure case by removing the cross-element terms in the sum of $|z_m + z_{\hat{m}}|^2$ and by replacing $(1-Q)^2 \mapsto (1-Q)$. This result is used throughout the remaining derivations.

5. RANDOM PHASE VARIATIONS

The second stochastic variation under consideration is random phase. Such variations represent either a byproduct of the elements and their associated electronics or a component of random positional variations. The former is modeled as a normally distributed (Gaussian) random phase variable with zero mean and a specified variance. The latter is modeled as a phase variation of the form $(\vec{k}_i - \vec{k}_s) \cdot \vec{\Delta x}_m$, where $\vec{\Delta x}_m$ is a Gaussian random positional variation, with random amplitude and direction. Because of the potential for very complicated probability density functions for the phase variation, a result for general phase variations is presented. The only restrictions on these variations are that they are statistically independent and identically distributed over the elements m.

Beginning with a phase variation ϕ_m that enters the beamforming expression for the element failures shown in equation (15) yields

$$V_2(\omega) = \sum_{m=1}^{M/2} \beta_m \left(z_m e^{i\phi_m} + z_{\hat{m}} e^{i\phi_{\hat{m}}} \right), \tag{26}$$

where the subscript 2 denotes element failures plus phase variations and β_m is defined as before. Letting the variable γ_m be defined by

$$\gamma_m = \exp(i\phi_m)\,, \tag{27}$$

the expression is rewritten as

$$V_2(\omega) = \sum_{m=1}^{M/2} \beta_m \left(\gamma_m z_m + \gamma_{\hat{m}} z_{\hat{m}} \right). \tag{28}$$

The stochastic moments of the variable γ_m are given by

$$E\{\gamma_m^k\} = \int \gamma_m^k p\,(\phi_m) d\phi_m\,, \qquad k > 0\,, \tag{29}$$

where $p\,(\phi_m)$ is the probability density function of the random phase variable ϕ_m. To maintain generality, the only assumptions for the specific form of the random variable ϕ_m are that ϕ_m and ϕ_n are independent and identically distributed for all $m \neq n$, so that $E\{\gamma_m^k\} = E\{\gamma_n^k\}$ for all $k > 0$ and all m, n.

As for the case of element failure, the expected value of the beamformer output power spectrum is evaluated as

$$E\{|V_2|^2\} = \sum_{m=1}^{M/2} \left(E\{|\psi_m|^2\} - |E\{\psi_m\}|^2 \right) + |E\{V_2\}|^2\,, \tag{30}$$

where

$$\psi_m = \beta_m(\gamma_m z_m + \gamma_{\hat{m}} z_{\hat{m}})$$

and a dependence on ω is assumed.

To evaluate the terms in equation (30), two properties of γ_m are derived. The first is given by

$$|\gamma_m|^2 = \gamma_m \gamma_m^* = \exp(i\phi_m)\exp(-i\phi_m) = 1\,, \tag{31}$$

which is independent of the random variable ϕ_m. The second property is

$$E\{\gamma_m \gamma_n^*\} = E\{\gamma_m\}E\{\gamma_n^*\} = E\{\gamma_m\}\left(E\{\gamma_n\}\right)^* = |E\{\gamma_m\}|^2\,, \tag{32}$$

which follows from γ_n and γ_m being independent and identically distributed. The two properties, shown in equations (31) and (32), are used to simplify the expressions in equation (30).

The first term in expression (30) is given by

$$\begin{aligned} E\{|\psi_m|^2\} &= E\{|\beta_m(\gamma_m z_m + \gamma_{\hat{m}} z_{\hat{m}})|^2\}\,, \\ &= E\{|\beta_m|^2\} \cdot E\{|\gamma_m|^2|z_m|^2 + |\gamma_{\hat{m}}|^2|z_{\hat{m}}|^2 + \gamma_m\gamma_{\hat{m}}^* z_m z_{\hat{m}}^* + \gamma_m^*\gamma_{\hat{m}} z_m^* z_{\hat{m}}\}\,, \\ &= (1-Q)^2 \cdot \left(|z_m|^2 + |z_{\hat{m}}|^2 + |E\{\gamma_m\}|^2(z_m^* z_{\hat{m}} + z_m z_{\hat{m}}^*)\right)\,; \end{aligned} \tag{33}$$

the second term is given by

$$\begin{aligned} |E\{\psi_m\}|^2 &= |E\{\beta_m(\gamma_m z_m + \gamma_{\hat{m}} z_{\hat{m}})\}|^2\,, \\ &= |E\{\beta_m\}|^2 \cdot |E\{\gamma_m z_m + \gamma_{\hat{m}} z_{\hat{m}})\}|^2\,, \\ &= (1-Q)^4 \cdot |E\{\gamma_m\}(z_m + z_{\hat{m}})|^2\,, \\ &= (1-Q)^4 \cdot |E\{\gamma_m\}|^2 \cdot |z_m + z_{\hat{m}}|^2\,; \end{aligned} \tag{34}$$

and the final term is given by

$$\begin{aligned} |E\{V_2\}|^2 &= |E\{\textstyle\sum \beta_m\,(\gamma_m z_m + \gamma_{\hat{m}} z_{\hat{m}})\}|^2\,, \\ &= |\textstyle\sum E\{\beta_m\}E\{\gamma_m z_m + \gamma_{\hat{m}} z_{\hat{m}}\}|^2\,, \\ &= |\textstyle\sum (1-Q)^2 \cdot (E\{\gamma_m\}z_m + E\{\gamma_{\hat{m}}\}z_{\hat{m}})|^2\,, \\ &= |(1-Q)^2 E\{\gamma_m\}\textstyle\sum(z_m + z_{\hat{m}})|^2\,, \\ &= (1-Q)^4 \cdot |E\{\gamma_m\}|^2 \cdot |V_0|^2\,, \end{aligned} \tag{35}$$

which makes use of the independence of $E\{\gamma_m\}$ on m. Combining these expressions back into equation (30) and then performing some algebra yields

$$\begin{aligned} E\{|V_2|^2\} &= \left[(1-Q)^2 - G^2(1-Q)^4\right] \sum_{m=1}^{M} w_m^2 \, d_m^2(\vec{\zeta}_i) \\ &+ \left[G^2(1-Q)^2 - G^2(1-Q)^4\right] \sum_{m=1}^{M/2} (z_m z_{\hat{m}}^* + z_m^* z_{\hat{m}}) \\ &+ G^2(1-Q)^4 \, |V_0|^2, \end{aligned} \tag{36}$$

where

$$G = |E\{\gamma_m\}| = |E\{e^{i\phi_m}\}|$$

is written to illustrate the independence on m.

Three notable limits are found in equation (36). First, when there are no phase variations, the expression becomes

$$E\{|V_2|^2\} \rightarrow \left[(1-Q)^2 - (1-Q)^4\right] \sum_{m=1}^{M/2} |z_m + z_{\hat{m}}|^2 + (1-Q)^4 |V_0|^2, \quad \text{as} \quad G \rightarrow 1 \, , \tag{37}$$

which, as expected, is the same case found in equation (23). Secondly, when the phase variations become completely unpredictable (i.e., $p(\phi_m) = 1/(2\pi)$ for $-\pi \leq \phi_m < \pi$), the impact is that $G \rightarrow 0$. In that case, the mean of the beamformer output power spectrum, shown in equation (36), becomes

$$E\{|V_2|^2\} \rightarrow (1-Q)^2 \sum_{m=1}^{M} w_m^2 \, d_m^2(\vec{\zeta}_i) \, , \qquad \text{as} \qquad G \rightarrow 0 \, , \tag{38}$$

which is the power output of an unphased (amplitude summation) array with random compensated failures. Finally, when there are no element failures ($Q \rightarrow 0$) but there are still phase perturbations, the mean beamformer output power spectrum becomes

$$E\{|V_2|^2\} \rightarrow (1-G^2) \sum_{m=1}^{M} w_m^2 \, d_m^2(\vec{\zeta}_i) + G^2 |V_0|^2 \, , \qquad \text{as} \qquad Q \rightarrow 0 \, , \tag{39}$$

which is the mean for an array with beamformer output spectrum

$$V(\omega) = \sum_{m=1}^{M} \gamma_m z_m \, , \tag{40}$$

as shown by Nuttall.[4]

As for the case of element failure, the expected value for the case of random phase variations with uncompensated element failures follows from equation (36), and is derived by replacing $(Q-1)^2 \mapsto (Q-1)$ and removing the cross-element terms from the element summations to yield

$$
\begin{aligned}
E\{|V_{2,uncomp}|^2\} &= \left[(1-Q) - G^2\,(1-Q)^2\right] \sum_{m=1}^{M} w_m^2\, d_m^2(\vec{\zeta}_i) \\
&\quad + G^2\,(1-Q)^2\,|V_0|^2 .
\end{aligned}
\tag{41}
$$

Comparing this expression with that in equation (36) shows the effect of element failure compensation on mean beamformer power output when random phase variations are included.

A combination of different physical effects may cause random phase errors. In that case, the net phase error ϕ_m at each element is written as a sum of the phase errors $\phi_m^{(j)}$ from the various physical effects. Thus, J independent physical effects causing a phase error produce

$$\phi_m = \sum_{j=1}^{J} \phi_m^{(j)} , \tag{42}$$

which results in

$$
\begin{aligned}
G &= |E\{\gamma_m\}| , \\
&= |E\{\exp(\textstyle\sum i\phi_m^{(j)})\}| , \\
&= \textstyle\prod |E\{\exp(i\phi_m^{(j)})\}| , \\
&= \textstyle\prod G^{(j)} ,
\end{aligned}
\tag{43}
$$

where $G^{(j)}$ is the value of G corresponding to the j^{th} source of the phase error. This expression follows from the statistical independence of the various sources of phase error.

6. RANDOM AMPLITUDE VARIATIONS

The final stochastic variation under consideration is random amplitude. The derivation begins by building upon equation (26) through the addition of a random amplitude variation term α_m. The nominal (no amplitude variation) case has $\alpha_m \to 1$, which is appropriate because no assumptions have yet been made on the specific statistical nature of the random variables in this study (i.e., they need not have zero mean). With the amplitude variations α_m, the beamformer output spectrum becomes

$$V_3(\omega) = \sum_{m=1}^{M/2} \beta_m \left(\alpha_m \gamma_m z_m + \alpha_{\hat{m}} \gamma_{\hat{m}} z_{\hat{m}}\right) , \tag{44}$$

where the subscript 3 denotes element failures plus phase and amplitude variations, and β_m, γ_m are defined as before. In the analysis that follows, the only assumption made for the random variable α_m is that it is statistically independent of m. Later examined is the special case where α_m has unit mean, which represents the physically important example of unbiased amplitude variations.

As in the previous cases, the expected value of the beamformer output power spectrum is evaluated as

$$E\{|V_3|^2\} = \sum_{m=1}^{M/2} \left(E\{|\lambda_m|^2\} - |E\{\lambda_m\}|^2\right) + |E\{V_3\}|^2 , \tag{45}$$

where

$$\lambda_m = \beta_m(\alpha_m \gamma_m z_m + \alpha_{\hat{m}} \gamma_{\hat{m}} z_{\hat{m}})$$

and a dependence on ω is assumed.

The first term in expression (45) is given by

$$\begin{aligned}
E\{|\lambda_m|^2\} &= E\{|\beta_m(\alpha_m \gamma_m z_m + \alpha_{\hat{m}} \gamma_{\hat{m}} z_{\hat{m}})|^2\} , \\
&= E\{\beta_m^2\} \cdot E\{\alpha_m^2 |\gamma_m|^2 |z_m|^2 + \alpha_{\hat{m}}^2 |\gamma_{\hat{m}}|^2 |z_{\hat{m}}|^2 \\
&\quad + \alpha_m \alpha_{\hat{m}} (\gamma_m \gamma_{\hat{m}}^* z_m z_{\hat{m}}^* + \gamma_m^* \gamma_{\hat{m}} z_m^* z_{\hat{m}})\} , \\
&= (1-Q)^2 \cdot \Big(E\{\alpha_m^2\} \cdot (|z_m|^2 + |z_{\hat{m}}|^2) \\
&\quad + \ E\{\alpha_m \alpha_{\hat{m}}\} \cdot |E\{\gamma_m\}|^2 \cdot (z_m^* z_{\hat{m}} + z_m z_{\hat{m}}^*)\Big) , \\
&= (1-Q)^2 \cdot E\{\alpha_m^2\} \cdot (|z_m|^2 + |z_{\hat{m}}|^2) \\
&\quad + (1-Q)^2 \cdot (E\{\alpha_m\})^2 \cdot |E\{\gamma_m\}|^2 \cdot (z_m^* z_{\hat{m}} + z_m z_{\hat{m}}^*) ;
\end{aligned} \tag{46}$$

the second term is given by

$$\begin{aligned}
|E\{\lambda_m\}|^2 &= |E\{\beta_m(\alpha_m\gamma_m z_m + \alpha_{\hat{m}}\gamma_{\hat{m}} z_{\hat{m}})\}|^2\,, \\
&= |E\{\beta_m\}E\{\alpha_m\}E\{\gamma_m\}(z_m + z_{\hat{m}})|^2\,, \\
&= |E\{\beta_m\}|^2 \cdot |E\{\alpha_m\}|^2 \cdot |E\{\gamma_m\}|^2 \cdot |z_m + z_{\hat{m}}|^2\,, \\
&= (1-Q)^4 \cdot (E\{\alpha_m\})^2 \cdot |E\{\gamma_m\}|^2 \cdot |z_m + z_{\hat{m}}|^2\,;
\end{aligned} \tag{47}$$

and the final term is given by

$$\begin{aligned}
|E\{V_3\}|^2 &= |E\{\textstyle\sum \beta_m(\alpha_m\gamma_m z_m + \alpha_{\hat{m}}\gamma_{\hat{m}} z_{\hat{m}})\}|^2\,, \\
&= |\textstyle\sum E\{\beta_m\} \cdot (E\{\alpha_m\gamma_m\} z_m + E\{\alpha_{\hat{m}}\gamma_{\hat{m}}\} z_{\hat{m}})|^2\,, \\
&= |(1-Q)^2 E\{\alpha_m\}E\{\gamma_m\} \textstyle\sum (z_m + z_{\hat{m}})|^2\,, \\
&= (1-Q)^4 \cdot (E\{\alpha_m\})^2 \cdot |E\{\gamma_m\}|^2 \cdot |V_0|^2\,.
\end{aligned} \tag{48}$$

For notational convenience, the variable A_k is defined as the k^{th} statistical moment of α_m, so that

$$A_k = E\{\alpha_m^k\} = \int \alpha_m^k p(\alpha_m) d\alpha_m\,, \qquad k > 0\,, \tag{49}$$

which is independent of m. Use of this simplification along with the previously defined G allows combination of the above terms into equation (45) and (after algebraic simplification) arrival at

$$\begin{aligned}
E\{|V_3|^2\} &= [\,A_2\,(1-Q)^2 - A_1^2\,G^2\,(1-Q)^4\,] \sum_{m=1}^{M} w_m^2\, d_m^2(\vec{\zeta}_i) \\
&+ [\,A_1^2\,G^2\,(1-Q)^2 - A_1^2\,G^2\,(1-Q)^4\,] \sum_{m=1}^{M/2} (z_m z_{\hat{m}}^* + z_m^* z_{\hat{m}}) \\
&+ A_1^2\,G^2\,(1-Q)^4\,|V_0|^2\,.
\end{aligned} \tag{50}$$

When there are no amplitude variations, $\alpha_m \equiv 1$, so that $A_1 = A_2 = 1$. In this case, the mean beamformer output power spectrum of equation (50) reduces to that shown in equation (36), as expected. Equation (50) represents the mean beamformer output power spectrum for compensated element failures, element phase errors, and element amplitude errors combined. It shows the important interrelations between the three different types of errors, thus justifying a thorough analysis of the combined effects.

A physically important class of amplitude variations is the *unbiased* amplitude variation, where $A_1 = E\{\alpha_m\} = 1$, but nothing can be said about $A_2 = E\{\alpha_m^2\}$. In this case, equation (50) reduces to

$$E\{|V_3|^2\} = \left[A_2\,(1-Q)^2 - G^2\,(1-Q)^4\right] \sum_{m=1}^{M} w_m^2\, d_m^2(\vec{\zeta}_i)$$

$$+ \left[G^2 (1-Q)^2 - G^2 (1-Q)^4\right] \sum_{m=1}^{M/2} (z_m z^*_{\hat{m}} + z^*_m z_{\hat{m}})$$

$$+ G^2 (1-Q)^4 |V_0|^2, \tag{51}$$

which shows the separation of amplitude and phase errors (G and A_2 never occur in the same term).

As for the previous cases, the results are compared with those found for uncompensated failures. As before, uncompensated failures are represented by replacing $(1-Q)^2 \mapsto (1-Q)$ and removing the cross-element terms from the element summations, which yields

$$\begin{aligned} E\{|V_{3,uncomp}|^2\} &= [A_2 (1-Q) - A_1^2 G^2 (1-Q)^2] \sum_{m=1}^{M} w_m^2 \, d_m^2(\vec{\zeta}_i) \\ &\quad + A_1^2 G^2 (1-Q)^2 |V_0|^2 \end{aligned} \tag{52}$$

for the general case with uncompensated failures and

$$\begin{aligned} E\{|V_{3,uncomp}|^2\} &= [A_2 (1-Q) - G^2 (1-Q)^2] \sum_{m=1}^{M} w_m^2 \, d_m^2(\vec{\zeta}_i) \\ &\quad + G^2 (1-Q)^2 |V_0|^2 \end{aligned} \tag{53}$$

for the case of unbiased amplitude errors with uncompensated failures.

7. SUMMARY OF VARIATIONS

The combined effects of element failures, element failure compensation, random phase variations, and random amplitude variations have been shown in the preceding sections. In all of the derived expressions, the use of an omnidirectional sensor directivity response and no error compensation reduces the results to those previously reported.[3,4] The detailed derivations have been presented to illustrate the level of approximation and assumptions involved in the development of the mean beampatterns. In this section, the principal results of the derivations are gathered for easy reference, and some additional comments are made with regard to their use in practice.

The two primary equations used for array error analysis are equation (52) for uncompensated failures and equation (50) for compensated failures, both of which are repeated here for convenience. For the general case of uncompensated failures, the mean beamformer output power spectrum is given by

$$\begin{aligned} E\{|V_{uncomp}|^2\} &= [\,A_2\,(1-Q) - A_1^2\,G^2\,(1-Q)^2\,]\sum_{m=1}^{M} w_m^2\, d_m^2(\vec{\zeta}_i) \\ &\quad + A_1^2\,G^2\,(1-Q)^2\,|V_0|^2\,, \end{aligned} \tag{54}$$

and for the general case of compensated failures, the mean beamformer output power spectrum is given by

$$\begin{aligned} E\{|V_{comp}|^2\} &= [\,A_2\,(1-Q)^2 - A_1^2\,G^2\,(1-Q)^4\,]\sum_{m=1}^{M} w_m^2\, d_m^2(\vec{\zeta}_i) \\ &\quad + [\,A_1^2\,G^2\,(1-Q)^2 - A_1^2\,G^2\,(1-Q)^4\,]\sum_{m=1}^{M} z_m^* z_{\hat{m}} \\ &\quad + A_1^2\,G^2\,(1-Q)^4\,|V_0|^2\,, \end{aligned} \tag{55}$$

where the compensation is any form of element blanking that is symmetric to failed elements. The second summation in expression (55) has been simplified from the previous form (see equation (50)) using the simplified relationship

$$\sum_{m=1}^{M/2} z_m z_{\hat{m}}^* = \sum_{m=M/2+1}^{M} z_m^* z_{\hat{m}}\,,$$

which follows from the definition of $z_{\hat{m}}$. The parameters Q and G in equations (54) and (55) represent the failure rate and the amplitude of the mean error due to phase perturbations, respectively. The parameters A_1 and A_2 represent the first and second moments of amplitude errors, respectively.

In these expressions, the nominal (no error) values of the parameters are $Q = 0$, $G = 1$, $A_1 = 1$, and $A_2 = 1$. Multiple independent phase errors with independent G's given by $G^{(j)}$ are

combined to have $G = \prod G^{(j)}$. Similarly, multiple independent amplitude errors with independent A_1's and A_2's are combined to have $A_1 = \prod A_1^{(j)}$ and $A_2 = \prod A_2^{(j)}$. Nondirective sensor elements are handled by modeling the sensor directivity as omnidirectional; hence, $d_m(\vec{\zeta}_i) \equiv 1$ in the above expressions provides equivalent results for omnidirectional sensors.

The first term in equations (54) and (55) shows that there is a sidelobe floor (minimal sidelobe level) due to errors that exists regardless of the level of sidelobe control for the nominal array. Because of the dependence on the incident signal through element directivity $d_m(\vec{\zeta}_i)$, it is difficult to quantify this effect for the general case. However, through the use of *directive array gain*, which is given by

$$AG = 10\log_{10}\left[\left(\sum w_m d_m(\vec{\zeta}_s)\right)^2\right] - 10\log_{10}\left[\sum w_m^2 d_m^2(\vec{\zeta}_i)\right] ,$$

an approximate sidelobe floor can be derived. In this expression, the first term is based on the element response in the steering direction $\vec{\zeta}_s$, and the second term is based on the incident direction $\vec{\zeta}_i$. Hence, the directive array gain expression AG is dependent on both the incident and steered directions.

For the case of uncompensated failures, the approximate sidelobe floor is given (in decibels) by

$$\min\{SL_{uncomp}\} \approx 10\log_{10}\left[\frac{A_2}{A_1^2 G^2(1-Q)} - 1\right] - AG . \tag{56}$$

As expected, improved array gain lowers the level of this sidelobe floor, even in the presence of errors in the array. For the case of compensated failures, the level of approximation applied to this sidelobe analysis is similar to removing the cross term in equation (55), in which case the approximate sidelobe floor is given by

$$\min\{SL_{comp}\} \approx 10\log_{10}\left[\frac{A_2}{A_1^2 G^2(1-Q)^2} - 1\right] - AG , \tag{57}$$

which is the same as for uncompensated failures with the working element rate $(1-Q)$ replaced with $(1-Q)^2$. Note that this method does not have the same effect as doubling the failure rate; in general, it produces a very small difference with a slightly better (lower) sidelobe level for the compensated failures as compared with the doubled failure rate. This behavior is the result of symmetry that is maintained because of the particular form of failure compensation employed. An example of the utility of these sidelobe floor approximations is given in the next section.

8. SAMPLE ERRORS

To accurately model the effects of various errors on the array, the impact of different combinations of G^2, A_1, A_2, and Q needs to be examined with regard to a particular array. These are the fundamental *beamformer error parameters* for beamforming an array of directional sensors. They provide the basic mathematical mechanism by which statistical errors enter the expressions for the statistical beampattern. In this set of parameters, G^2 replaces G since that is the only form in which G enters the mean beampattern expressions.

In the remaining sections, the derivation of these parameters is presented for common types of errors. The physical mechanisms that affect the parameters are translated into effective values for beamformer errors. To assess the effect of the parameter values on specific beampatterns, the following numerical example is considered: a 20-by-20 square planar array of elements spaced at Nyquist sampling ($\lambda/2$ spacing between elements), where each element has a cosine directivity pattern in the direction normal to the array surface. The array is shaded in both directions with Taylor shading[5] to a sidelobe level of 25 dB with $\bar{n} = 4$. The beampattern for this array under perfect conditions (no errors and no failures) and steered to the array maximum response axis is given in figure 1 (only a slice of the beampattern through one direction is shown due to symmetry). Figure 2 shows the effect of 10% and 20% failures on the beampattern ($Q = 0.10$ and $Q = 0.20$, respectively), both without compensation and with diagonal blanking compensation. In figure 3, the effects of phase errors on array response are shown for values of $G^2 = 0.80, 0.90$, and 0.95, and in figure 4, the effects of unbiased amplitude errors ($A_1 = 1.00$) are shown for values of $A_2 = 1.05, 1.10, 1.15$, and 1.20. Figure 5 illustrates the combined effects of a set of phase and amplitude errors with both uncompensated and compensated failures. In this case, the simulation was performed with $G^2 = 0.90$, $A_1 = 1.00$, $A_2 = 1.10$, and $Q = 0.10$. For combined errors, the nulls of the sidelobes are much less significant than for any of the component errors. Thus, the combination of errors both increases the peak sidelobe level (through addition) and smooths out the oscillations in the sidelobe response. This smoothing effect is more significant with large errors than with small errors.

The utility of expressions (56) and (57) for approximating the sidelobe floor can be shown by considering the combined error case in figure 5. For uncompensated failures, the sidelobe floor expression in equation (56) was used to obtain the approximate floor shown in figure 6, which allows an assessment of the improved shading in the array. When a new shading scheme is considered to lower sidelobes, two effects must be addressed. The first is that the floor will decrease according to the increase in directive array gain (AG). However, if this decrease does not bring the sidelobe floor below the level of desired sidelobe control, the shading becomes error dominated, which is an undesirable condition that should be avoided. If conservative error estimates are known at the beamformer design stage, the shading may be adjusted to manage this sidelobe control versus sidelobe floor tradeoff, so that the result will minimize oscillations in the sidelobes of the beampattern.

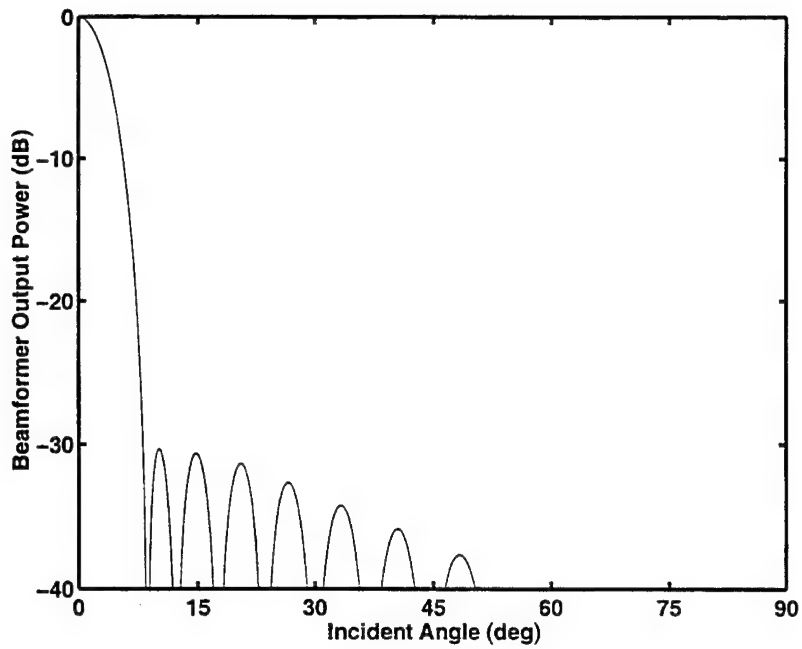

Figure 1. Nominal Beampattern for the 20-by-20 Array

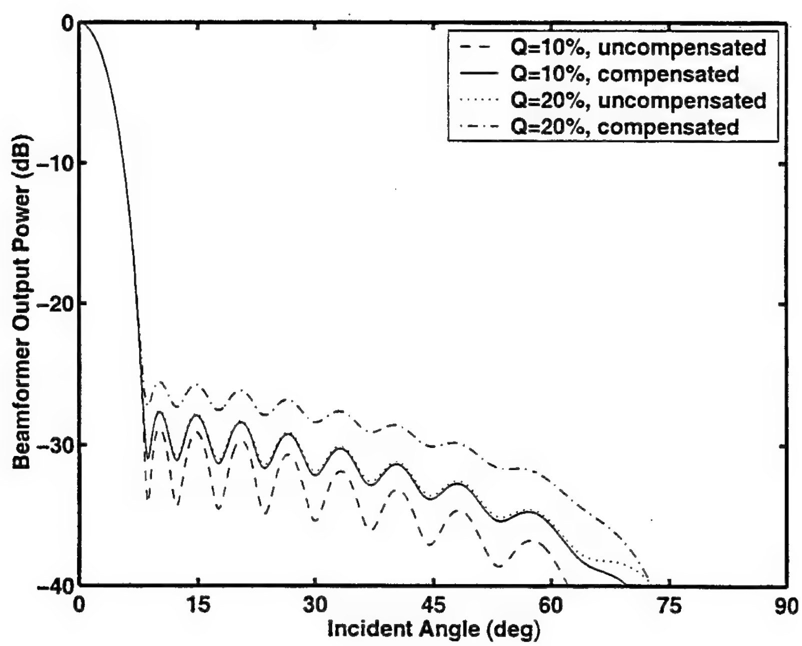

Figure 2. Beampattern for the Array with Failures

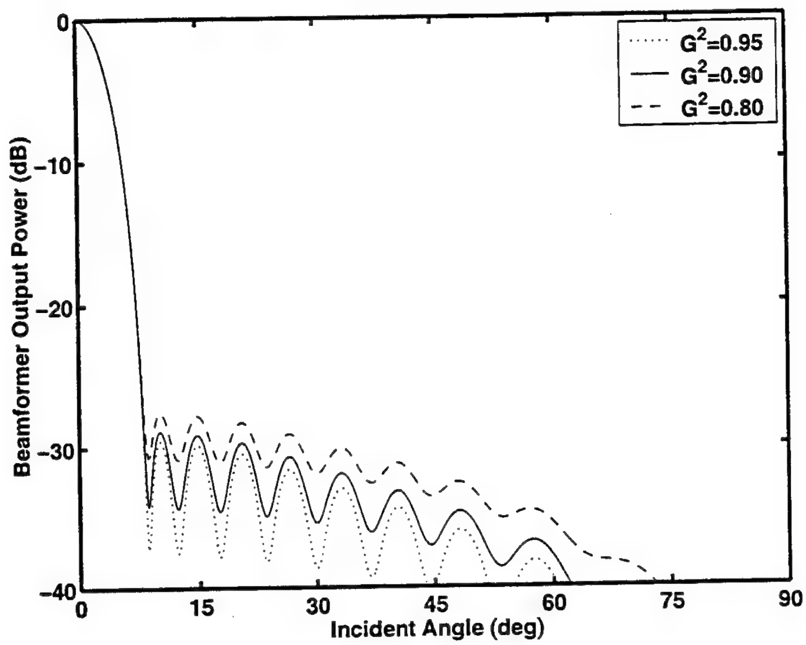

Figure 3. Beampattern for the Array with Phase Errors of Varying G^2

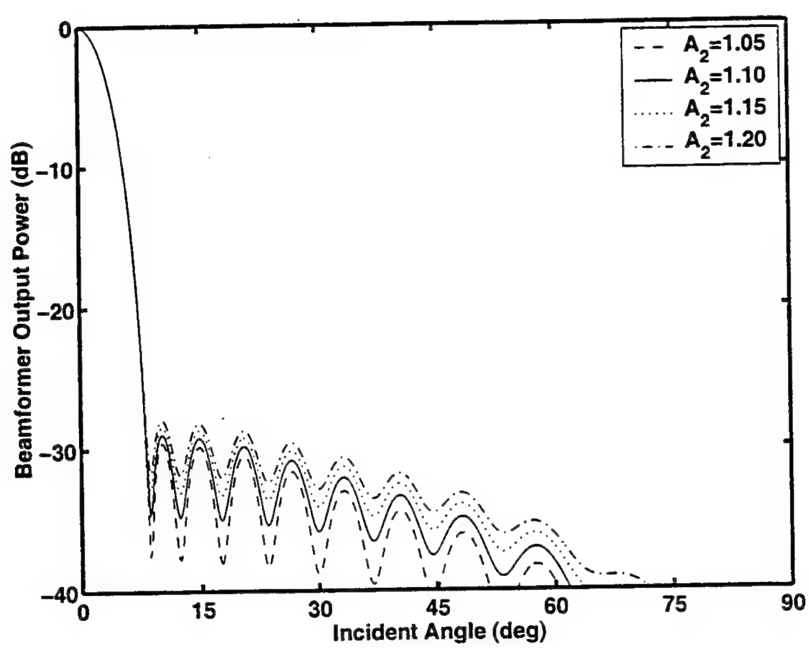

Figure 4. Beampattern for the Array with Amplitude Errors of Varying A_2

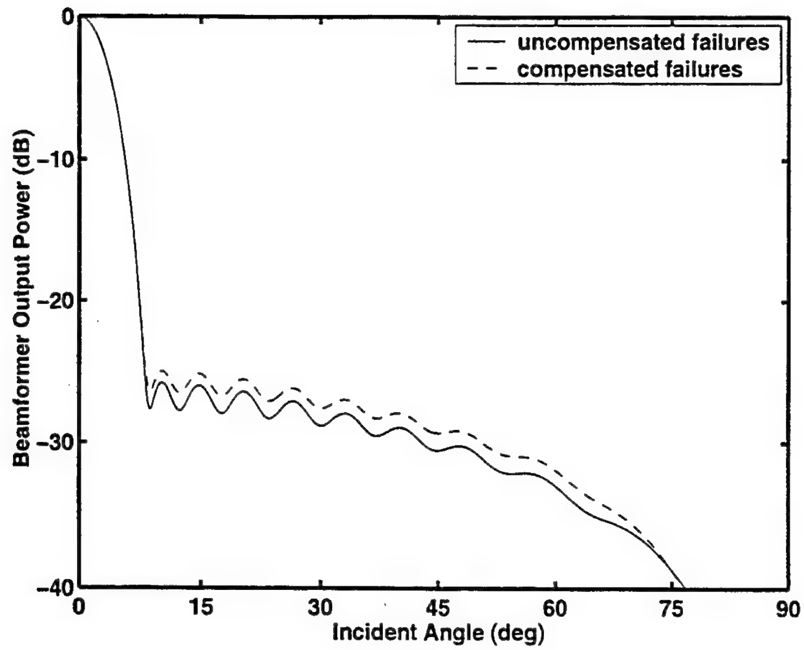

Figure 5. Beampattern for the Array with $Q = 0.10$*,* $G^2 = 0.90$*,* $A_1 = 1.00$*, and* $A_2 = 1.10$

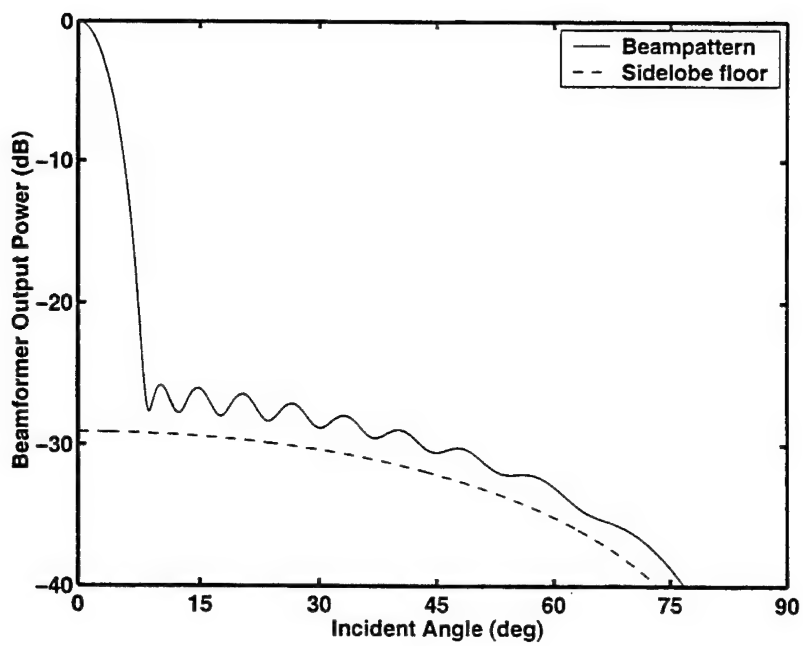

Figure 6. Sidelobe Floor for Combined Errors in Uncompensated Case

8.1 NOMINAL PHASE ERRORS

As an example of phase variation, consider the simple case where ϕ_m is a zero-mean Gaussian variable with standard deviation σ_p (variance σ_p^2) so that

$$p(\phi_m) = \frac{1}{\sigma_p\sqrt{2\pi}} \exp\left(\frac{-\phi_m^2}{2\sigma_p^2}\right). \tag{58}$$

One use of such a model may be to represent expected variations in element response due to electronics. In this case, the term G in equation (36) is found by evaluating

$$\begin{aligned} E\{\gamma_m\} &= \int \gamma_m p(\phi_m) d\phi_m\,, \\ &= \frac{1}{\sigma_p\sqrt{2\pi}} \int \exp(i\phi_m) \exp\left(\frac{-\phi_m^2}{2\sigma_p^2}\right) d\phi_m\,, \\ &= \frac{1}{\sigma_p\sqrt{2\pi}} \int \exp\left(-\frac{(\phi_m - i\sigma_p^2)^2}{2\sigma_p^2} - \frac{\sigma_p^2}{2}\right) d\phi_m\,, \\ &= \exp\left(-\frac{\sigma_p^2}{2}\right). \end{aligned} \tag{59}$$

The term G in equation (36) is thus given by

$$G = |E\{\gamma_m\}| = \exp\left(\frac{-\sigma_p^2}{2}\right). \tag{60}$$

8.2 ELEMENT POSITION ERRORS

Element position errors are an important type of phase error. In this case, the individual element phase is given by

$$\phi_m^{nom} = (\vec{k}_i - \vec{k}_s) \cdot \vec{x}_m \tag{61}$$

when there is no position error. It is assumed that the beamformer contains no knowledge of a positional error and therefore applies phase delays according to the assumed (nominal) position. Thus, the element positional errors only affect the $\vec{k}_i \cdot \vec{x}_m$ component of phase. In this situation, the phase is modified to

$$\phi_m^{pert} = \vec{k}_i \cdot (\vec{x}_m + \Delta\vec{x}) - \vec{k}_s \cdot \vec{x}_m = (\vec{k}_i - \vec{k}_s) \cdot \vec{x}_m + \vec{k}_i \cdot \Delta\vec{x}\,, \tag{62}$$

with an error in position $\Delta\vec{x}$.

The perturbation in this case consists of a deterministic vector $\vec{k}_i$ (with unspecified direction and known magnitude) dotted with a random positional error vector (with arbitrary direction and

random magnitude). The error in position can therefore be stipulated to be a stochastic magnitude with an arbitrary direction, resulting in two independent random variables: a magnitude $M = |\Delta\vec{x}|$ and an angle θ that is the angle between the incident wavevector $\vec{k}_i$ and the displacement vector $\Delta\vec{x}$. Thus, the dot product of the error term in the above expression can be rewritten as

$$\phi_m^{err} = \phi_m^{pert} - \phi_m^{nom} = \vec{k}_i \cdot \Delta\vec{x} = (\omega/c)M\cos(\theta)\,. \tag{63}$$

Because the error in displacement vector can be located in any direction (in three-dimensional space) with equal probability and the direction of the beamformer steering component is fixed in a statistical sense (i.e., it is deterministic), the combination $M\cos(\theta)$ is a statistical variable that covers all angles in $0 \le \theta \le \pi$ equally for each non-negative value of M. The physical meaning of M and θ is replaced with statistically identical variables such that M is normally (Gaussian) distributed about zero and θ is uniformly distributed as an angle over the upper hemisphere of a unit sphere (it is arbitrarily made to be the angle relative to the north pole). These variables still apply in the same way to ϕ_m^{err}, but the use of a normally distributed M (allowing values less than zero) facilitates the solution. An alternative approach with an identical solution is to assume M to be Rayleigh distributed, with θ uniformly distributed as an angle over the entire surface of a unit sphere.

Because the angle θ is a random variable with uniform distribution over the upper hemisphere of a unit sphere (relative to the north pole), it has a cumulative distribution function $c(\theta_0) = Prob(\theta \le \theta_0)$ given by (with $\theta \in \{0, \pi/2\}$)

$$c(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\theta_0} \cos(\theta)\; d\theta\; d\phi = \sin(\theta_0)\,, \qquad \theta_0 \le \pi/2\,. \tag{64}$$

The probability density function of the random variable θ is given by

$$p(\theta) = \frac{d}{d\theta}c(\theta) = \cos(\theta)\,, \qquad 0 \le \theta \le \pi/2\,. \tag{65}$$

The random variable M is assumed to be normally (Gaussian) distributed with zero mean and a standard deviation given by σ_M. Thus,

$$p(M) = \frac{1}{\sqrt{2\pi}\sigma_M}\exp\left(\frac{-M^2}{2\sigma_M^2}\right) \tag{66}$$

is the probability density function of the random variable M. Note that this variable has an equal probability of being positive or negative, so that the resulting statistics of the combination of $M \cdot \cos(\theta)$ covers a statistically uniform sphere with random magnitude. The distribution of the random magnitude is given by the right-half (greater than zero) side of the normal distribution in equation (66).

The function G needed for the mean beamformer output power spectrum under phase errors is evaluated by forming

$$
\begin{aligned}
E[\gamma_m] &= E[e^{i\phi_m^{err}}], \\
&= \int_{-\infty}^{\infty}\int_{0}^{\pi/2} e^{i\phi_m^{err}} p(\theta)p(M)d\theta dM, \\
&= \frac{1}{\sqrt{2\pi}\sigma_M}\int_{-\infty}^{\infty}\int_{0}^{\pi/2} \exp\left(\frac{iM\omega\cos(\theta)}{c} - \frac{M^2}{2\sigma_M^2}\right)\cos(\theta)d\theta dM. \qquad (67)
\end{aligned}
$$

A change of variables from M to nondimensional $\hat{M} = (\omega/c)M$ and from θ to nondimensional $\eta = \cos(\theta)$ is used to reduce the double integral to

$$
\begin{aligned}
E[\gamma_m] &= \frac{-c}{\omega\sqrt{2\pi}\sigma_M}\int_{-\infty}^{\infty}\int_{1}^{0} \exp\left(i\hat{M}\eta - \frac{\hat{M}^2c^2}{2\omega^2\sigma_M^2}\right)\frac{\eta}{\sqrt{1-\eta^2}}\, d\eta\, d\hat{M}, \\
&= \frac{c}{\omega\sqrt{2\pi}\sigma_M}\int_{0}^{1}\frac{\eta}{\sqrt{1-\eta^2}}\int_{-\infty}^{\infty}\exp\left(i\hat{M}\eta - \frac{\hat{M}^2c^2}{2\omega^2\sigma_M^2}\right)\, d\hat{M}\, d\eta, \\
&= \int_{0}^{1}\frac{\eta}{\sqrt{1-\eta^2}}\exp\left(\frac{-\omega^2\sigma_M^2\eta^2}{2c^2}\right)\, d\eta, \\
&= \frac{-i\sqrt{2\pi}c}{2\omega\sigma_M}\exp\left(\frac{-\omega^2\sigma_M^2}{2c^2}\right)\,\mathrm{erf}\left(\frac{i\omega\sigma_M}{\sqrt{2}c}\right), \qquad (68)
\end{aligned}
$$

where the function $\mathrm{erf}(x)$ is the error function with argument x. The above expression uses no analytical assumptions and is exact. In practice, the standard deviation σ_M of the magnitude of the error is assumed to be small compared with a wavelength. Thus,

$$\sigma_M \ll c/f,$$

which implies

$$\sigma_M\omega/(\sqrt{2}c) < \varepsilon$$

for some small ε. The assumption of small σ_M (compared with a wavelength) is a realistic one, because position error magnitudes on the order of a wavelength would render a phase-based acoustic array useless. Since the argument of the error function is now of small magnitude, the small argument error function expansion (derived from equation 7.1.5 in Abramowitz and Stegun[6])

$$\mathrm{erf}(ix) = \frac{2i}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)} \qquad (69)$$

is used to arrive at (after some algebra)

$$G = |E[\gamma_m]| = e^{-S} \sum_{n=0}^{\infty} \frac{S^n}{n!(2n+1)}, \tag{70}$$

where

$$S = \frac{\omega^2 \sigma_M^2}{2c^2}$$

is a nondimensional scaling of the standard deviation of position error magnitude. In practice, the summation in this expression may be truncated after a few terms (two or three terms is usually enough) for small position errors.

It is useful to compare the phase error due to positional errors with the highly conservative approximation to phase error that is often employed. Restating the error in position as three independent errors in orthogonal directions (say x, y, and z for convenience) creates a phase error of

$$\phi_m^{err} = \phi_m^{pert} - \phi_m^{nom} = k_i^x \Delta x^x + k_i^y \Delta x^y + k_i^z \Delta x^z, \tag{71}$$

where superscripts are used to represent the coordinate directions. The random positional error is now modeled with each component $(\Delta x^x, \Delta x^y, \Delta x^z)$ as an independent, zero-mean normally (Gaussian) distributed random variable with standard deviation σ_M. The approach followed is to bound each component of $\vec{k}_i$ by its magnitude $k_0 = \omega/c$, which creates a conservative bound on the phase error of

$$|\phi_m^{err}| \leq 3\, k_0\, |D|, \tag{72}$$

where D is a random variable that is distributed identically to the components of $\Delta\vec{x}$. Using the nominal phase error result of the previous section, G is now given by

$$G = \exp\left(\frac{-9\,\omega^2\sigma_M^2}{2c^2}\right) = \exp(-9\,S), \tag{73}$$

where S is defined as before. This approach is identical to computing a bound on an independent G for each coordinate direction and then multiplying these bounds together. A comparison of this very conservative bound and the exact term (as computed in equation (70)) is shown in figure 7.

A second (and more accurate) conservative approach is developed by creating the phase error in the form

$$\phi_m^{err} = k_i^x \Delta x^x + k_i^y \Delta x^y + k_i^z \Delta x^z = k_0\, |D|\, (\tilde{k}_i \cdot \Delta\tilde{x}), \tag{74}$$

where $\tilde{k}_i$ and $\Delta\tilde{x}$ are the unit vectors in the same direction as $\vec{k}_i$ and $\Delta\vec{x}$, respectively. The dot-product occurring in the preceding expression is now conservatively bounded in the interval $[-1, 1]$, such that the phase error is conservatively estimated by

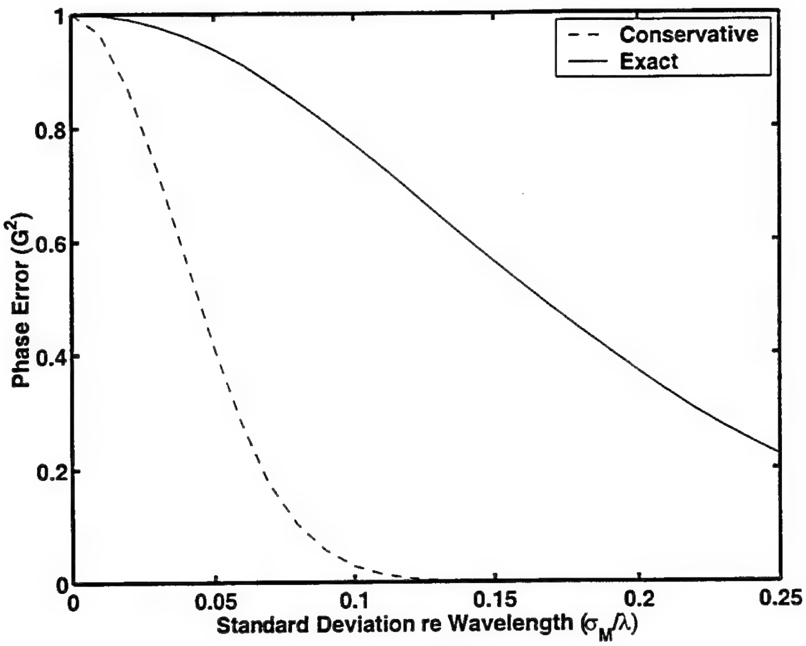

Figure 7. Phase Error G^2 for Highly Conservative Error Approximation

$$|\phi_m^{err}| \leq k_0\,|D|\,, \tag{75}$$

which is obviously less conservative than the previous bound. This estimate leads to

$$G \;=\; \exp\left(\frac{-\omega^2\sigma_M^2}{2c^2}\right) \;=\; \exp(-S)\,, \tag{76}$$

with the same parameter S. A comparison of this less conservative bound and the exact term (as computed in equation (70)) is shown in figure 8.

From figures 7 and 8, it is clear that the addition of separate bounds on the different directional components of error causes a very large deviation from the full statistical error. This result is due to the misrepresentation of $\vec{k}_i$ as three independent vectors of equal magnitude. A better conservative bound is the error bound in equation (75), which is based upon a bound on the dot-product of the incident wavevector and the element perturbation vector. Although it is not recommended to use this bound (since a more exact solution in the form of equation (70) is already available), it provides some practical guidance for using nonuniform element perturbations. If the perturbations in different coordinate directions vary in magnitude (such as when the x-position is more tightly controlled than y and z), then a reasonable first estimate of G can be obtained by taking

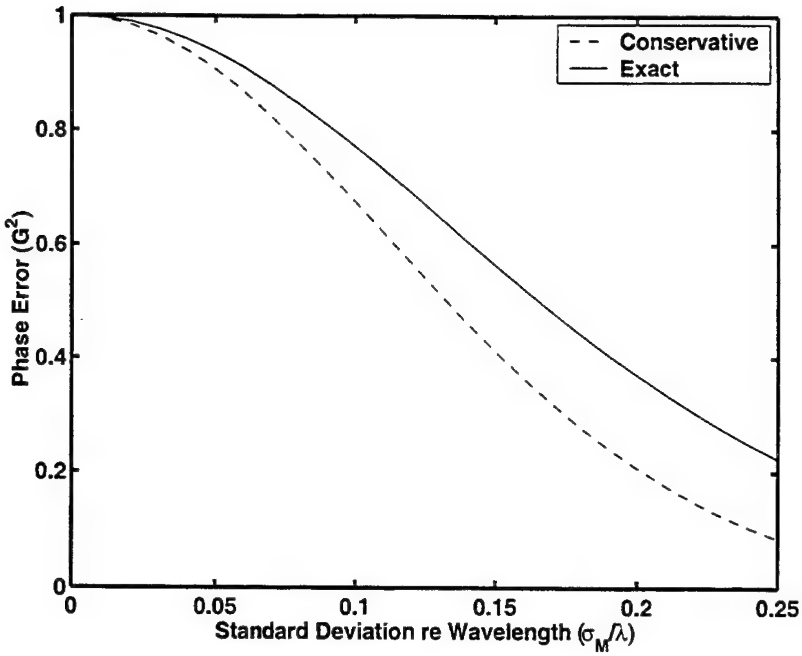

Figure 8. Phase Error G^2 for Less Conservative Error Approximation

$$|\phi_m^{err}| \leq k_0 \cdot \max D \,, \tag{77}$$

where $\max D$ is a random variable representing the deviation in the dimension with the maximum perturbation (highest standard deviation). This approach provides better estimates than the one that independently adds conservative estimates for error due to each direction (with separate, but identical, k_0 on each term).

The preceding method uses the equivalent phase error approach outlined earlier. A more accurate solution can be obtained by applying the perturbation to each incident angle and independently computing the resulting beamformer output at that angle; however, that level of detailed modeling eliminates any insight that can be gained from the first-order effects models.

8.3 ELEMENT ORIENTATION ERRORS

Element orientation errors are a type of error that produces closed-form approximate statistical results in specialized cases. For the analysis here, attention is restricted to cosine-directive elements with perfect baffling. In this case, the orientation affects the beampattern through the element directivity term $d_m(\vec{\zeta_i}) = \max\{\vec{\zeta_i} \cdot \vec{n}_m, 0\}$, where $\vec{\zeta_i}$ is the unit vector in the incident wave direction and $\vec{n}_m$ is the unit vector normal to the element. The effect of

the element orientation error is a resulting random perturbation in the normal vector that creates a magnitude change in the element directivity. The random perturbation can be represented as

$$d_m(\vec{\zeta}_i) = \max\{\vec{\zeta}_i \cdot (\vec{n}_m + \Delta\vec{n}),\ 0\} = \max\{(\vec{\zeta}_i \cdot \vec{n}_m + \vec{\zeta}_i \cdot \Delta\vec{n}),\ 0\}\,, \tag{78}$$

where $\Delta\vec{n}$ is the perturbation to the orientation vector. Since the incident direction $\vec{\zeta}_i$ is fixed (in a statistical sense), the perturbation term $\vec{\zeta}_i \cdot \Delta\vec{n}$ represents a random magnitude multiplied by the cosine of a random angle. In the form given, the perturbation is limited to the constraint $|\vec{n}_m + \Delta\vec{n}| = 1$, which makes the connection between the dot-products and the angle cosine meaningful.

The constraint $|\vec{n}_m + \Delta\vec{n}| = 1$ has the following geometrical interpretation: the vector $\vec{n}_m$ is given as a unit vector (by definition), such that any sum of this vector and another vector that is added to the unit vector must lie on the unit sphere whose origin is the same as the origin of $\vec{n}_m$. Some geometrical analyses show that this is only possible when the added vector $\Delta\vec{n}$ has a magnitude given by

$$|\Delta\vec{n}| = 2\sin\left(\frac{\phi_m}{2}\right), \tag{79}$$

where ϕ_m is the angle between $\vec{n}_m$ and $\vec{n}_m + \Delta\vec{n}$. The variable ϕ_m now has a physical interpretation as the angular error in orientation for the m^{th} element of the array. To this point, the interpretation of $|\Delta\vec{n}|$ and ϕ_m is independent of the specific element directivity of the array being analyzed.

Since the perturbation $\vec{\zeta}_i \cdot \Delta\vec{n}$ has a magnitude given by $|\Delta\vec{n}|$, the only remaining component to analyze is the random phase. The phase of $\vec{\zeta}_i \cdot \Delta\vec{n}$ is analyzed by first recognizing that *for small perturbations* $\Delta\vec{n}$ lies nearly on the plane normal to the vector $\vec{n}_m$ (as a result of the sum $\vec{n}_m + \Delta\vec{n}$ lying on the same unit sphere as $\vec{n}_m$, and $|\Delta\vec{n}|$ being small). Defining the angle ψ_m as the angle between $\vec{n}_m$ and $\vec{\zeta}_i$ yields

$$\cos\psi_m = \vec{n}_m \cdot \vec{\zeta}_i\,. \tag{80}$$

The angle φ is defined as the angle between $\Delta\vec{n}$ and $\vec{\zeta}_i$ such that

$$\cos\varphi = \frac{\Delta\vec{n} \cdot \vec{\zeta}_i}{|\Delta\vec{n}|}\,. \tag{81}$$

A final angle θ is defined as the rotation of the perturbation vector $\Delta\vec{n}$ in the plane normal to the vector $\vec{n}_m$, so that θ is the random variable for the direction of the orientation error for the element. Since θ is a random variable with uniform distribution, the point $\theta = 0$ is arbitrary and can be set to be the location where the projection of $\vec{\zeta}_i$ intersects the plane normal to $\vec{n}_m$. Some simple geometry shows that the following relationship holds:

$$\cos\varphi = \sin\psi_m \cos\theta\,. \tag{82}$$

Combining equation (79) with equations (81) and (82) yields

$$\vec{\zeta}_i \cdot \Delta\vec{n} = 2\sin\left(\frac{\phi_m}{2}\right)\sin\psi_m \cos\theta \tag{83}$$

as an expression for the perturbation under small orientation errors. Two further approximations simplify this formula. First, the assumption of small orientation errors implies that $|\phi_m| < \varepsilon$ where $0 < \varepsilon \ll 1$, so that $2\sin(\phi_m/2) \approx \phi_m$. Secondly, directive elements whose normal is far from the incident wave direction contribute little to the beampattern, which implies that $|\psi_m| < \varepsilon$ for terms that contribute significantly to the beamformer output. Under this assumption, $\sin\psi_m \approx \psi_m \cos\psi_m$, which holds for the elements that are the primary contributors to the beampattern. The elements that are the most significant contributors in the unperturbed situation remain the most significant under small perturbation. With these simplifications, equation (83) reduces to

$$\begin{aligned}\vec{\zeta}_i \cdot \Delta\vec{n} &\approx \phi_m\psi_m \cos\psi_m \cos\theta\,,\\ &\approx (\phi_m\psi_m\cos\theta)(\vec{\zeta}_i \cdot \vec{n}_m)\,.\end{aligned} \tag{84}$$

By restricting the analysis to cosine directive elements with perfect baffling, equation (78) becomes

$$\begin{aligned}d_m(\vec{\zeta}_i) &= \max\{\alpha_m(\vec{\zeta}_i \cdot \vec{n}_m),\ 0\}\,,\\ &\approx \alpha_m \cdot \max\{\vec{\zeta}_i \cdot \vec{n}_m,\ 0\}\,,\end{aligned} \tag{85}$$

where the approximation holds for significant contributors to the beampattern (elements with nontrivial directivity) and

$$\alpha_m = 1 + \phi_m\psi_m\cos\theta \tag{86}$$

is used to represent the element magnitude perturbation due to orientation errors for cosine directive elements.

To use the statistical analysis developed in earlier sections, the random variables α_m (comprised of ϕ_m, θ, and ψ_m) must be independent and identically distributed. The variable ϕ_m represents orientation error magnitude (in radians), or tilt, and the variable θ is the direction of this tilt, so that these variables are assumed to be identically distributed. However, the variable ψ_m represents the angle between the incident vector and the vector normal to the unperturbed element, which is not random and necessarily varies among the elements. Since this variable must be identical for all elements, an approximation must be used. Two approximations have been tested: one sets ψ_m to be the average of all ψ_m's for each incident angle and the other sets ψ_m to the maximum ψ_m for each incident angle. Because the former is easier to implement and the difference in the two was seen to be minor,

$$\psi_m \approx \left(\frac{1}{M}\right) \sum_{j=1}^{M} \arccos(\vec{\zeta}_i \cdot \vec{n}_j)\,, \qquad . \, \forall\, m\,, \tag{87}$$

is a reasonable approximation for arrays with small curvature. Under this restriction, the problem is appropriate for the analysis developed in earlier sections.

The physical interpretation of the two random variables implies that θ is uniformly distributed over $\{-\pi, \pi\}$ and ϕ_m follows a distribution similar in shape to the right side (positive component only) of a normal (Gaussian) distribution with zero mean. To facilitate the analytical simplifications, the random variable θ is restricted to $\{0, \pi\}$ and ϕ_m is allowed to follow a true Gaussian distribution (permitting both positive and negative values) with zero mean and a stated standard deviation of σ_o. In this way, the variable ϕ_m is allowed to have values ranging from $\{-\infty, \infty\}$ even though the values outside of $\{-\pi, \pi\}$ are nonphysical. But for small perturbations, these values are on the extreme tails of the distribution and therefore are negligible. Thus, σ_o is the standard deviation of element orientation (tilt) with respect to normal, given in radians, and

$$p(\phi_m) = \frac{1}{\sigma_o\sqrt{2\pi}} \exp\left(\frac{-\phi_m^2}{2\sigma_o^2}\right). \tag{88}$$

The necessary statistics on α_m are given by

$$\begin{aligned}
A_1 &= E[\alpha_m]\,, \\
&= \int_{-\infty}^{+\infty} \int_0^{\pi} (1 + \phi_m \psi_m \cos\theta)\, p(\phi_m)\, p(\theta)\, d\theta\, d\phi_m\,, \\
&= \frac{1}{\pi\sigma_o\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^{\pi} (1 + \phi_m \psi_m \cos\theta) \exp\left(\frac{-\phi_m^2}{2\sigma_o^2}\right) d\theta\, d\phi_m\,, \\
&= \frac{1}{\sigma_o\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-\phi_m^2}{2\sigma_o^2}\right) d\phi_m\,, \\
&= 1\,,
\end{aligned} \tag{89}$$

which shows the perturbations to be unbiased (i.e., equation (51) holds), and

$$\begin{aligned}
A_2 &= E[\alpha_m^2]\,, \\
&= \int_{-\infty}^{+\infty} \int_0^{\pi} (1 + \phi_m \psi_m \cos\theta)^2\, p(\phi_m)\, p(\theta)\, d\theta\, d\phi_m\,, \\
&= \frac{1}{\pi\sigma_o\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^{\pi} (1 + \phi_m \psi_m \cos\theta)^2 \exp\left(\frac{-\phi_m^2}{2\sigma_o^2}\right) d\theta\, d\phi_m\,, \\
&= \frac{1}{2\sigma_o\sqrt{2\pi}} \int_{-\infty}^{+\infty} (2 + \phi_m^2 \psi_m^2) \exp\left(\frac{-\phi_m^2}{2\sigma_o^2}\right) d\phi_m\,, \\
&= 1 + \frac{\psi_m^2 \sigma_o^2}{2}\,,
\end{aligned} \tag{90}$$

for which equation (87) is used to obtain the value of ψ_m for each incident angle.

8.4 ELEMENT RESPONSE TOLERANCES

Element response tolerances are a simple form of error that can be modeled by placing statistical variations on the magnitude of individual sensors. If these variations are assumed to be normally distributed about the mean, a simple normal (Gaussian) distribution is

$$p(\alpha_m) = \frac{1}{\sigma_\alpha \sqrt{2\pi}} \exp\left(\frac{-(\alpha_m - 1)^2}{2\sigma_\alpha^2}\right), \tag{91}$$

where α_m is the magnitude of the m^{th} sensor. In this statistical model, it is assumed that the nominal (mean) sensor response is $\alpha_m = 1$, and the standard deviation from this mean is given by σ_α. If element response tolerances are given as a standard deviation in received power (in decibels), then a transformation between α_m and β_m (the tolerance in decibels) shows

$$\alpha_m = 10^{(\beta_m/20)}, \tag{92}$$

where β_m is now a random variable with a mean of zero and a standard deviation of σ_β, which is the stated standard deviation in power level. For this case, the errors may have a bias in magnitude; thus, the parameter A_1 must be computed as

$$\begin{aligned} A_1 = E\{\alpha_m\} &= \int 10^{(\beta/20)} p(\beta)\, d\beta, \\ &= \frac{1}{\sigma_\beta\sqrt{2\pi}} \int 10^{(\beta/20)} \exp\left(\frac{-\beta^2}{2\sigma_\beta^2}\right) d\beta, \end{aligned} \tag{93}$$

which is best evaluated numerically. In a similar manner, the parameter A_2 is given by

$$A_2 = E\{\alpha_m^2\} = \frac{1}{\sigma_\beta\sqrt{2\pi}} \int 10^{(\beta/10)} \exp\left(\frac{-\beta^2}{2\sigma_\beta^2}\right) d\beta, \tag{94}$$

which is also best evaluated numerically.

The use of a normally distributed magnitude variation as in equation (91) can be easily examined analytically. In this case, the required statistical moments A_1 and A_2, as found in equation (50), are given by

$$A_1 = E\{\alpha_m\} = 1 \tag{95}$$

and

$$\begin{aligned} A_2 &= E\{\alpha_m^2\}, \\ &= E\{(\alpha_m - \mu_\alpha)^2\} + \mu_\alpha^2, \\ &= \sigma_\alpha^2 + 1, \end{aligned} \tag{96}$$

where $\mu_\alpha = E\{\alpha_m\} = A_1 = 1$ is the mean value and the third step follows from the definition of variance. These expressions can then be used in equation (50) to obtain the effects of element response tolerances on mean beamformer spectral power output.

9. SUMMARY

The error analysis of a two-dimensional passive array (conformal or planar) with directive element responses has been developed. The analysis was based on a set of fundamental beamformer error parameters that address effective element-level phase errors, amplitude errors, and statistical failures (reliability). The combination of these various parameters and their effect on mean beampattern power output was given. Also presented were numerical examples of the errors, as well as the analytical derivation of various physical error mechanisms in terms of amplitude and phase error. Because the mathematical derivations described here pertain to a wide variety of passive arrays (and include the effects of array curvature and element directivity with the errors), these expressions will have broad application for assessing the impact of element-level errors on array beamformer performance.

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